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ORIGINAL PAPER

Special Lie symmetry and Hojman conserved quantity of Appell equations for a Chetaev nonholonomic system

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Abstract A special Lie symmetry and Hojman conserved quantity of the Appell equations for a Chetaev nonholonomic system are studied. The differential equations of motion and Appell equations of the Chetaev nonholonomic system are established. Under the special Lie symmetry group transformations in which the time is invariable, the determining equation of the special Lie symmetry of the Appell equations for a Chetaev nonholonomic system is given, and the expression of the Hojman conserved quantity is deduced directly from the Lie symmetry. Finally, an example is given to illustrate the application of the results.

Keywords Appell equations · Chetaev nonholonomic system · Lie symmetry · Hojman conserved quantity

1 Introduction

Appell equations are very important equations in analytical mechanics, and belong to one of the three types of mechanical system in the theory of analytical mechanics [1]. Nearly 20 years, Chinese scholars have gained fruitful achievements in research, promotion and application of the Appell equation [2–8]. Since 2000, Chinese scholars have made some achievements in the studies of symmetry for mechanical system with constraints [9–24], especially in Lie symmetry's research [25–39]. However, for a long time, there have been fewer results to Appell equations. To solve Appell equations, Mei Feng-Xiang, the famous classical mechanics expert, first gained the Noether conserved quantity deduced indirectly from the Noether symmetry by Mei symmetry [40]; Ref. [41] examined the conserved quantity of the variable mass holonomic system deduced indirectly from the Noether symmetry by Mei symmetry; Ref. [4] obtained the conserved quantity of Appell equations for the rotational relativistic holonomic system deduced indirectly from the Noether symmetry and Lie symmetry by Mei symmetry; Ref. [42] used Appell to express the structure equation of Mei symmetry and Mei conserved quantity for a non-Chetaev's type constrained mechanical system. However, there are fewer research results of Lie symmetry of Appell equations. This paper works on special Lie symmetry and Hojman conserved quantity of Appell equations for a Chetaev type nonholonomic system.

2 Appell equations for a Chetaev nonholonomic system

Suppose that the position of a mechanical system is determined by the n generalized coordinates q_s ($s =$

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$1, 2, \ldots, n$), and it is subject to the g ideal bilateral Chetaev nonholonomic constraints

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \quad (\beta = 1, 2, \ldots, g). \tag{1}$$

The restriction condition of constraints imposed on virtual displacement is

$$\frac{\partial f_\beta}{\partial \dot{q}_s} \delta q_s = 0 \quad (\beta = 1, 2, \ldots, g). \tag{2}$$

The acceleration energy of system is $S = S(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$, the generalized forces are $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, constraint multipliers are $\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. Then the Appell equations of the system are

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}. \tag{3}$$

From Eqs. (1) and (3), all the constraint multipliers λ_β can be found, and Eq. (3) can be rewritten as

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \Lambda_s. \tag{4}$$

These are called equations of the holonomic system corresponding to the nonholonomic system (1) and (3), where

$$\Lambda_s = \Lambda_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{5}$$

are generalized constraint forces. It has been proved that the solution of the corresponding holonomic system (4) gives the motion of the nonholonomic system if the initial conditions of motion satisfy the constraint equation (1). The differential equations of motion of the system can be solved from Eq. (4):

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, 2, \ldots, n). \tag{6}$$

3 Determining equation and definition of Lie symmetry

Introducing the special infinitesimal transformations of group in which the time is invariable, a vector $\tilde{\boldsymbol{X}}^{(0)}$ of the infinitesimal generators and its first expansion $\tilde{\boldsymbol{X}}^{(1)}$ and its second expansion $\tilde{\boldsymbol{X}}^{(2)}$ along the trajectory of motion of the system are

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{7}$$

$$\tilde{\boldsymbol{X}}^{(0)} = \xi_s \frac{\partial}{\partial q_s}, \tag{8}$$

$$\tilde{\boldsymbol{X}}^{(1)} = \tilde{\boldsymbol{X}}^{(0)} + \frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_s}, \tag{9}$$

$$\tilde{\boldsymbol{X}}^{(2)} = \tilde{\boldsymbol{X}}^{(1)} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} \frac{\partial}{\partial \ddot{q}_s}. \tag{10}$$

where [40]

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \alpha_s \frac{\partial}{\partial \dot{q}_s} + \dot{\alpha}_s \frac{\partial}{\partial \ddot{q}_s}. \tag{11}$$

ε denotes an infinitesimal parameter in Eq. (7), ξ_s are infinitesimal generators from Eq. (7) to Eq. (10).

Notice that $\frac{\partial S}{\partial \ddot{q}_s} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s}$; the determining equation of Lie symmetry of Appell equations (4) for a Chetaev nonholonomic system can be written as

$$\tilde{\boldsymbol{X}}^{(2)} \left(\frac{\partial S}{\partial \ddot{q}_s} \right) = \tilde{\boldsymbol{X}}^{(1)}(Q_s) + \tilde{\boldsymbol{X}}^{(1)}(\Lambda_s). \tag{12}$$

The determining equation of Lie symmetry for (6) can be written as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s = \frac{\partial \alpha_s}{\partial q_k} \xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_k. \tag{13}$$

The restriction equation of Lie symmetry for nonholonomic constraints of Eq. (1) under the special infinitesimal transformations (7) can be expressed as

$$\tilde{\boldsymbol{X}}^{(1)} \left\{ f_\beta(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \right\} = 0. \tag{14}$$

It can be easily proved that restriction equations (2) of which the constraint equation (1) is imposed on the virtual displacement δq_s can be rewritten in the following form:

$$\frac{\partial f_\beta}{\partial \dot{q}_s} \xi_s = 0 \quad (\beta = 1, 2, \ldots, g; s = 1, 2, \ldots, n). \tag{15}$$

Equation (15) is called the additional restriction equation.

Definition 1 If the infinitesimal generators ξ_s satisfy the determining equations (12) or (13), then the relevant symmetry is the Lie symmetry of the holonomic system (4) or (6) corresponding to the Chetaev nonholonomic system (1) and (3).

Definition 2 If the infinitesimal generators ξ_s satisfies determining equations (12) or (13) and restriction equation (14), then the relevant symmetry is a weak Lie symmetry of the holonomic system (4) or (6) corresponding to the Chetaev nonholonomic system (1) and (3).

Definition 3 If the infinitesimal generators ξ_s satisfy the determining equations (12) or (13), restriction equation (14) and additional restriction equation (15), then the relevant symmetry is a strong Lie symmetry of the holonomic system (4) or (6) corresponding to the Chetaev nonholonomic system (1) and (3).

4 Hojman conserved quantity deduced from the Lie symmetry

Proposition 1 *If the infinitesimal generators ξ_s are generators of a Lie symmetry of the Chetaev nonholonomic system* (1) *and* (3), *and satisfy the determining equations* (12) *or* (13), *and if for a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ holds*

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{16}$$

then the Hojman conserved quantity deduced directly from the Lie symmetry of Appell equations for the corresponding holonomic system is

$$I_H = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu \xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const.} \tag{17}$$

Proposition 2 *If the infinitesimal generators ξ_s are generators of a weak Lie symmetry of the Chetaev nonholonomic system* (1) *and* (3), *and satisfy the determining equations* (12) *or* (13) *and restriction equation* (14), *and if a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies Eq.* (16), *then the Hojman conserved quantity deduced directly from a weak Lie symmetry of Appell equations for the corresponding holonomic system is Eq.* (17).

Proposition 3 *If the infinitesimal generators ξ_s are generators of strong Lie symmetry of the Chetaev nonholonomic system* (1) *and* (3), *and satisfy the determining equations* (12) *or* (13), *restriction equation* (14) *and additional restriction equation* (15), *and if a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies Eq.* (16), *then the Hojman conserved quantity deduced directly from a strong Lie symmetry of Appell equations for the corresponding holonomic system is Eq.* (17).

5 An illustrative example

In the following, we give an example, which only to illustrate the application of the above results.

A particle of quality as m moved in three-dimensional space, whose Appell function, generalized force and Chetaev's type nonholonomic constraint equations are

$$S = \frac{1}{2}m\left(\ddot{q}_1^2 + \ddot{q}_2^2 + \ddot{q}_3^2\right), \tag{18}$$

$$Q_1 = Q_2 = 0, \qquad Q_3 = -mg, \tag{19}$$

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0. \tag{20}$$

m and g are constants in Eqs. (18) and (19). We try to study the Lie symmetry and Hojman conserved quantity deduced directly from the Lie symmetry of the Appell equations for the system.

From Eq. (3), we get

$$\begin{aligned} &m\ddot{q}_1 = 2\lambda \dot{q}_1, \qquad m\ddot{q}_2 = 2\lambda \dot{q}_2, \\ &m\ddot{q}_3 = -mg - 2\lambda \dot{q}_3. \end{aligned} \tag{21}$$

From Eqs. (20) and (21), we obtain

$$\lambda = -\frac{mg}{4\dot{q}_3}. \tag{22}$$

By substituting Eq. (22) into Eq. (23), we obtain

$$\begin{aligned} &m\ddot{q}_1 = -\frac{mg\dot{q}_1}{2\dot{q}_3}, \qquad m\ddot{q}_2 = -\frac{mg\dot{q}_2}{2\dot{q}_3}, \\ &m\ddot{q}_3 = -\frac{mg}{2}. \end{aligned} \tag{23}$$

The determining equation (13) of the Lie symmetry gives

$$\begin{aligned} &\ddot{\xi}_1 = \dot{\xi}_1\left(-\frac{mg}{2\dot{q}_3}\right) + \dot{\xi}_3 \frac{mg\dot{q}_1}{2\dot{q}_3^2}, \\ &\ddot{\xi}_2 = \dot{\xi}_2\left(-\frac{mg}{2\dot{q}_3}\right) + \dot{\xi}_3 \frac{mg\dot{q}_2}{2\dot{q}_3^2}, \\ &\ddot{\xi}_3 = 0. \end{aligned} \tag{24}$$

According to the restriction equation (14), we can obtain

$$\dot{q}_1\dot{\xi}_1 + \dot{q}_2\dot{\xi}_2 - \dot{q}_3\dot{\xi}_3 = 0. \tag{25}$$

From the additional restriction equation (15), we get

$$\dot{q}_1\xi_1 + \dot{q}_2\xi_2 - \dot{q}_3\xi_3 = 0. \tag{26}$$

The above-mentioned equations (24) have the following solutions:

$$\xi_1 = \dot{q}_1, \qquad \xi_2 = \dot{q}_2, \qquad \xi_3 = \dot{q}_3, \tag{27}$$

$$\xi_1 = \xi_2 = \xi_3 = 1, \tag{28}$$

$$\xi_1 = \xi_2 = 0, \qquad \xi_3 = \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2gq_3\right)^2. \tag{29}$$

Apparently, the infinitesimal generators (27) are generators of strong Lie symmetry, which satisfy Eqs. (24)–(26), and Eqs. (28) and (29) are generators of weak Lie symmetry, which satisfy Eqs. (24) and (25).

Furthermore, from Eq. (16), we obtain

$$-\frac{g}{\dot{q}_3} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln \mu = 0. \tag{30}$$

The above-mentioned equation (30) has the following solutions:

$$\mu = \frac{1}{\dot{q}_3^2}, \tag{31}$$

$$\mu = \frac{1}{\dot{q}_3^2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2gq_3\right). \tag{32}$$

By using the proposition, from Eqs. (28) and (31), we have a Hojman conserved quantity

$$I_{H_1} = 0. \tag{33}$$

From Eqs. (28) and (32), we obtain

$$I_{H_2} = 2g\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2gq_3\right)^{-1} = \text{const.} \tag{34}$$

Equations (29) and (31) give

$$I_{H_3} = 4g\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2gq_3\right) = \text{const.} \tag{35}$$

From Eqs. (29) and (32), we obtain

$$I_{H_4} = 6g\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2gq_3\right) = \text{const.} \tag{36}$$

From Propositions 1–3, we know that $I_{H_1}, I_{H_2}, I_{H_3}$ and I_{H_4} are Hojman conserved quantities deduced directly from the weak Lie symmetry of the Appell equations for the Chetaev nonholonomic system, where I_{H_1} is the usual conserved quantity. The conserved quantities with obvious physical meanings can be found by using Newtonian mechanics, but there are not plenty; more conserved quantities can be found by using the analytical mechanics methods, but the physical meanings of some are often not obvious. However, we can find more and more conserved quantities by the symmetry theory, but physical meaning of some conserved quantities is less obvious. It is a problem of suspense that still remains to be resolved, when we find the conserved quantities by using the symmetry theory.

6 Conclusion

The Hojman conserved quantity deduced directly from a special Lie symmetry of the Appell equations for the Chetaev nonholonomic system is gained from this paper. The results of this paper can further spread to the studies of Lie symmetry of Appell equations for non-Chetaev nonholonomic system and nonholonomic system of non-Chetaev's type with unilateral constraints. There are few studies about Lie symmetry of Appell equation, so the results of this paper have much significance in perfecting and developing the theory of Lie symmetry and conserved quantity for the mechanical system.

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